nation with hydrogen. This has been specifically considered with major fuel properties such as sulfur content, smoke point and thermal stability.

The extensive degree of sulfur removal by a cobalt molybdate catalyst and aromatics saturation by a noble metal catalyst have been cited. This plus the effect of temperature, pressure and other reactor variables have illustrated the flexibility of the process, which gives uniquely high yields of quality product without a waste disposal problem.

Engineering Notes

ENGINEERING NOTES are short manuscripts describing new developments or important results of a preliminary nature. These Notes cannot exceed 6 manuscript pages and 3 figures; a page of text may be substituted for a figure and vice versa. After informal review by the editors, they may be published within a few months of the date of receipt. Style requirements are the same as for regular contributions (see inside back cover).

Flutter Test Analysis in the Time Domain Using a Recursive System Representation

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Introduction

FLUTTER test analyses usually involved determining estimates of modal frequency and damping for the system under consideration and then following trends of these results with speed and Mach number to check for a possible flutter condition. Developments in this field are aimed at reducing test and analysis time and improving the accuracy of results in the presence of noise (i.e., turbulence).

Recently there have been interesting developments toward rapid analysis with two methods, the "Least Squares Identification" method[1,2] and the "Correlation Fit" method, described for the first time in this Note. They extract the results directly and automatically from the time histories without resort to the frequency domain and the analysis of vector plots. Both these methods involve choosing a model for the unknown system, representing it recursively,[3] and matching it to the data by curve fitting. It is believed that their noise rejection should be at least as good as that for other methods such as the Autocorrelation and Crosscorrelation methods.[4]

The Least Squares Identification method fits the assumed model directly to the time histories by the method of least squares.[5] In this Note, the method is examined and shown to be equivalent to an exact fit to the first few lag values of the autocorrelation of response and the crosscorrelation of excitation and response. The need for decimation, mentioned in Ref. 1, is explained.

The Correlation Fit method also is described. This is believed to be new and is still at an early stage in development. A "model match" is carried out by writing the autocorrelation and cross correlation functions in terms of the unknown model coefficients and carrying out a simultaneous least squares fit to a chosen number of lag values of these correlation functions from which the coefficients are determined. Thus the Least Squares Identification method is shown to be a special case of the Correlation Fit method.

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Basic Test and Analysis Procedure

The system of interest (such as an aircraft) is excited with any known input, though the methods can be adapted easily to cope with an unknown input. A sinusoidal frequency sweep is a suitable input since it tends to separate the modes. The response is measured and the signals sampled. Normally, the procedure is to choose segments of these data, each containing only one or two dominant modal response peaks.

A model transfer function is assumed for the system within each segment and a hold[3] is included. Using Z transform theory,[1-3] a recursive relationship between sampled response and excitation (y_j and x_j, $j = 1,2 \ldots L$) can be written in the form

$$y_j = \sum_{i=1}^{N} a_i y_{j-i} + \sum_{i=0}^{M} b_i x_{j-i} \tag{1}$$

where a_i and b_i are the coefficients defining the denominator and numerator of the model Z transfer function and hence the modal frequencies and dampings.

With no noise and a correctly chosen model, Eq. (1) is satisfied exactly by the measured values and therefore can be solved for the coefficients. However, where noise is present or the model does not correspond to the real system, the equation can be thought of either as giving an estimated response $\hat{y}_j$ different from the measured value y_j or as an approximate relationship between measured values. Thus an estimate of the coefficients can be obtained from this equation through a curve fit or model match, carried out in different ways by the two methods.

Least Squares Identification Method

If a segment that starts at point NS and finishes at NF is selected from the L point measured excitation and response records, and the general Nth order model in Eq. (1) is chosen, then the procedure[1] is to write this equation for each point in matrix form as

$$\begin{Bmatrix} y_{NS} \\ y_{NS+1} \\ \vdots \\ y_{NF} \end{Bmatrix} = \begin{bmatrix} y_{NS-1} & \cdots & y_{NS-N} & x_{NS} & \cdots & x_{NS-M} \\ y_{NS} & \cdots & y_{NS-N+1} & x_{NS+1} & \cdots & x_{NS-M+1} \\ \vdots & & \vdots & \vdots & & \vdots \\ y_{NF-1} & \cdots & y_{NF-N} & x_{NF} & \cdots & x_{NF-M} \end{bmatrix} \begin{Bmatrix} a_1 \\ \vdots \\ a_N \\ b_0 \\ \vdots \\ b_M \end{Bmatrix} \tag{2}$$

or $\mathbf{Y} = S\mathbf{A}$ (NS is chosen such that y_{NS-N} and x_{NS-M} are known). Equation (2) is solved, using the method of least squares, by premultiplying with S^T,

$$S^T\mathbf{Y} = S^TS\mathbf{A}$$

and therefore

$$\mathbf{A} = (S^TS)^{-1}S^T\mathbf{Y} \tag{3}$$

i.e., there are effectively ($NF - NS + 1$) sets of data that are related, apart from noise and modeling errors, by Eq. (1). Equations (2) and (3) show a least squares fit to these data, yielding estimates of the model coefficients and hence the modal frequency and damping.

If this method is written in a different form, new understanding can be gained. The general form of the above least squares fit can be established from

$$\underset{ai,bi}{\text{MIN}}\ J$$

where J is a "cost" function given by

$$J = \sum_{j=NS}^{NF} (y_j - \hat{y}_j)^2$$

or

$$J = \sum_{j=NS}^{NF}\left(y_j - \sum_{i=1}^{N} a_i y_{j-i} - \sum_{i=0}^{M} b_i x_{j-i}\right)^2 \tag{4}$$

By using $\partial J/\partial b_k = 0$ and $\partial J/\partial a_k = 0$, then after some manipulation

$$\sum_{j=NS-k}^{NF-k} x_j y_{j+k} = \sum_{i=1}^{N} a_i\left\{\sum_{j=NS-k}^{NF-k} x_j y_{j+k-i}\right\} + \sum_{i=0}^{M} b_i\left\{\sum_{j=NS-k}^{NF-k} x_j x_{j+k-i}\right\};\quad k = 0, 1, 2 \ldots M \tag{5a}$$

$$\sum_{j=NS-k}^{NF-k} y_j y_{j+k} = \sum_{i=1}^{N} a_i\left\{\sum_{j=NS-k}^{NF-k} y_j y_{j+k-i}\right\} + \sum_{i=0}^{M} b_i\left\{\sum_{j=NS-k}^{NF-k} y_j x_{j+k-i}\right\};\quad k = 1, 2 \ldots N \tag{5b}$$

which can be seen to be equivalent to Eq. (3).

If the fit is carried out using the entire L point records and they are assumed to be transient (i.e., x_j and y_j are zero for $j = 0, -1, -2 \ldots$.) then the summation limits in Eq. (5) can be altered from $(NS - k) \rightarrow (NF - k)$ to $1 \rightarrow (L - k)$. All these summations then look like autocorrelations and crosscorrelations at different lags, for example, cross correlation of excitation and response at lag k, $R_{xy}(k) = \Sigma x_j y_{j+k}$; $k << L$. The left-hand summations in Eqs. (5) are exact correlations, but the right-hand ones are only approximate because of slight inconsistencies in the limits. Equations (5) then could be written as an approximate recursive correlation expression involving the first few lag values of each correlation. This thought leads to a new method, which will now be discussed.

Correlation Fit Method

If the sampled signals x_j and y_j ($j = 1,2 \ldots L$) are considered and again assumed to be transient, they are related by the discrete equivalent of the convolution integral or by the recursive expression (1). The cross correlation of excitation and response (R_{xy}) and the autocorrelation of response (R_{yy}) can be defined at lag k as

$$\left.\begin{aligned} R_{xy}(k) &= \frac{1}{L-k}\sum_{j=1}^{L-k} x_j y_{j+k} \quad &(8a)\\ R_{yy}(k) &= \frac{1}{L-k}\sum_{j=1}^{L-k} y_j y_{j+k} \quad &(8b)\end{aligned}\right\} k = 0, 1, 2 \ldots.$$

Using Eq. (1) to substitute for y_{j+k} in Eqs. (8) then, after reordering the summations and cancelling the factor $1/L - k$,

$$\sum_{j=1}^{L-k} x_j y_{j+k} = \sum_{i=1}^{N} a_i\left\{\sum_{j=1}^{L-k} x_j y_{j+k-i}\right\} + \sum_{i=0}^{M} b_i\left\{\sum_{j=1}^{L-k} x_j x_{j+k-i}\right\};\quad k = 0, 1, 2 \ldots (NC - 1) \tag{9a}$$

$$\sum_{j=1}^{L-k} y_j y_{j+k} = \sum_{i=1}^{N} a_i\left\{\sum_{j=1}^{L-k} y_j y_{j+k-i}\right\} + \sum_{i=0}^{M} b_i\left\{\sum_{j=1}^{L-k} y_j x_{j+k-i}\right\};\quad k = 0, 1, 2 \ldots (NA - 1) \tag{9b}$$

Thus the correlation functions at different lags have been expressed in terms of the coefficients of the assumed model. Again the equation is exact with no noise, and a correct model, but, when noise is present, it becomes approximate, and the right-hand sides can be thought of as giving estimates of correlations at lag k. The model match is then carried out by solving the simultaneous correlation Eqs. (9) for as many lags of the autocorrelation (NA) and crosscorrelation (NC) functions as desired using, for example, the method of least squares. The constraint on the choice of NA and NC is that $(NA + NC) \geq (M + N + 1)$. This method essentially involves curve fitting chosen parts of both the autocorrelation and cross correlation functions to yield the desired estimates, and it therefore has been called the Correlation Fit method.

It is useful to note the similarity of the above expressions to the well known relationships for *continuous* transient records (length T sec). The process carried out is the same as that when the convolution relation (exact for no noise!)

$$y(t) = \int_0^t h(\sigma)x(t-\sigma)d\sigma \qquad 0 \leqslant t \leqslant T \tag{10}$$

where $h(\sigma)$ is the system impulse response function, is used to substitute for $y(t + \tau)$ in the continuous correlation expressions for lag τ,

$$\left.\begin{aligned} R_{xy}(\tau) &= \frac{1}{T-\tau}\int_0^{T-\tau} x(t)y(t+\tau)dt \quad &(11a)\\ R_{yy}(\tau) &= \frac{1}{T-\tau}\int_0^{T-\tau} y(t)y(t+\tau)dt \quad &(11b)\end{aligned}\right\} \tau \geqslant 0$$

to give, after some manipulation of limits and orders of integration,

$$R_{xy}(\tau) = \int_0^{T+\tau} h(\sigma)R_{xx}(\tau - \sigma)d\sigma \tag{12a}$$

$$R_{yy}(\tau) = \int_0^{T+\tau} h(\sigma)R_{xy}(\sigma - \tau)d\sigma \tag{12b}$$

where R_{xx} is the autocorrelation of excitation.

Equation (12a) with, in general, a "white" excitation assumption is that used in the cross correlation method to give an estimate for $h(\sigma)$ whereas the autocorrelation expression (12b) is normally used in the form where (10) is used to substitute for both $y(t)$ and $y(t + \tau)$ in Eq. (11) to give,

$$R_{yy}(\tau) = \int h(\sigma_1)\int h(\sigma_2)R_{xx}(\tau - \sigma_2 + \sigma_1)d\sigma_2 d\sigma_1 \tag{13}$$

The autocorrelation method also assumes a "white" excitation spectrum such that Eq. (13) can be simplified to yield another estimate of frequency and damping.[4] Normally only one of these methods is used in a flutter test.

Therefore, it is clear that the Correlation Fit method involves essentially the same derivation process as used for the standard correlation methods without any assumption about the excitation. Therefore, it is basically a combination of the autocorrelation and crosscorrelation meth-

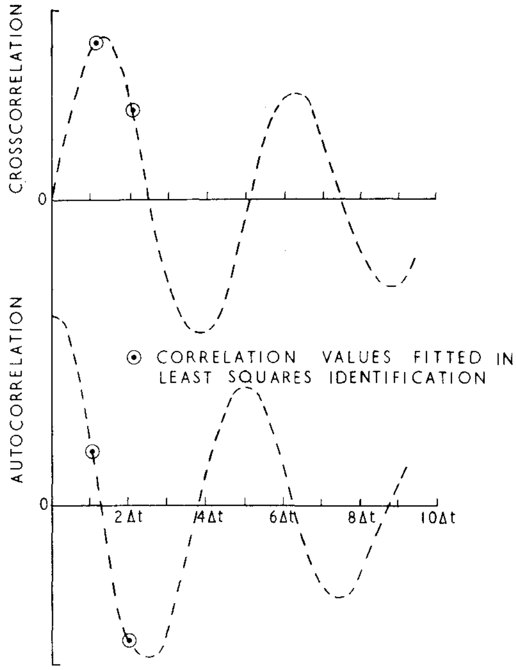

Fig. 1 Typical one-degree-of-freedom correlation functions showing values fitted in the Least Squares Identification method for a model of order $N = 2$ (n.b. displacement response, $b_0 = 0$).

ods where fewer lag values are used but where they are generally taken from the higher signal/noise part of the correlations. It is not advisable to use too many lag values since the correlation functions deteriorate in accuracy as the desired components decay. With no noise a fit of either correlation (i.e., $NA = 0$ or $NC = 0$) would yield correct answers but in the presence of noise the estimate found from the combined fit should be statistically better than that from one or other correlation alone.

So far what has been said applies to a model match carried out using correlations corresponding to the entire transient records. If, however, a segment of data is chosen, the summation limits in Eqs. (9) can be changed from $1 \rightarrow (L - k)$ to $NS \rightarrow (NF - k)$ with the equation so formed still being mathematically correct. Although strictly the definitions of correlation functions lose their meaning for a nontransient segment, pseudo-correlations can be defined (with limits $NS \rightarrow NF - k$ or possibly $NS - k \rightarrow NF - k$) to retain the concept of a correlation fit. Thus the method can still be used for a segment of data.

Discussion

Similarity of Methods

It is immediately evident that the basic expressions involved in the Correlation Fit and Least Squares Identification methods are *identical* when written for the entire transient records. This is also true for a segment of the data apart from the different limits introduced by the choice of the pseudo-correlation definition mentioned above. Thus both methods carry out the model match by curve fitting the two correlation functions. In the case of the Correlation Fit method it is a *least squares* fit using as many lag values as desired whereas for the Least Squares Identification method the fit is *exact* using the lag values dictated by the assumed model. The latter method is therefore a special case of the Correlation Fit method with $NA = N$ and $NC = M + 1$, even though the formulations are different.

For a one-degree-of-freedom example, the Least Squares Identification method fits only to two lag values of each

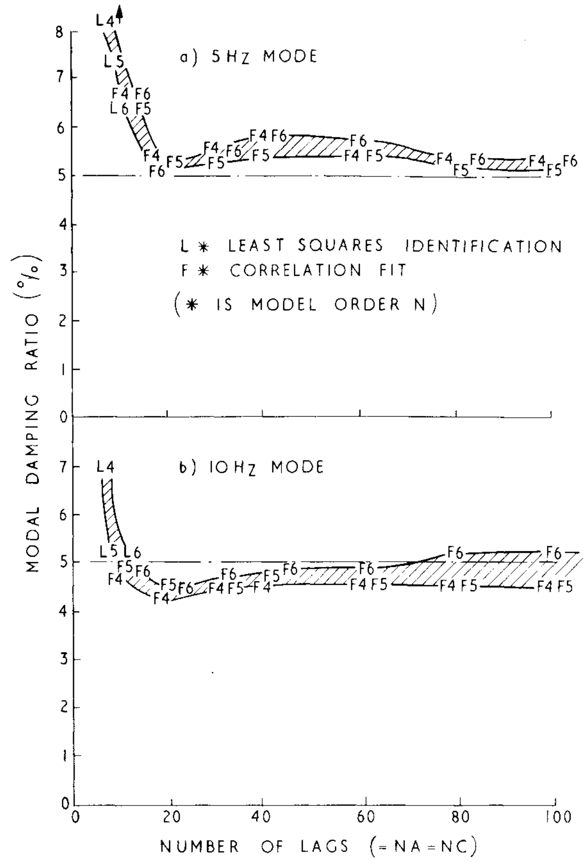

Fig. 2 Damping estimates from analysis of two-degree-of-freedom displacement response to sweep, with noise (5 and 10 Hz, both 5% critically damped, sweep rate 4.5 rad/sec², Δt = 0.01 sec).

correlation (Fig. 1) although it makes use of the approximate correlations on the right-hand side of Eq. (5). Since the standard correlation methods use many more lags, it seems surprising that this method gives such good answers, especially with noise present!

Decimation

Decimation involves resampling the data to reduce the number of points/cycle to approximately five at the high frequency end of the segment,[1] thereby increasing the sampling interval (Δt sec). In Ref. 1 it is claimed that decimation is carried out "to reduce computation time." However, this author believes that there is a more important reason and has found that decimation is sometimes necessary to obtain even sensible answers.

If the Least Squares Identification method is thought of in terms of correlations, the need for decimation becomes obvious. Since the correlations at lags of $\Delta t, 2\Delta t \ldots$ sec are fitted (Fig. 1), if the sampling interval were too small then the points used would be contained within too small a portion of the first correlogram cycle and the results would be very sensitive to any error. This was shown in a sensitivity analysis carried out for a two degree of freedom example in which the sensitivities were reduced very significantly by only doubling the sampling interval.

Realizing that the Least Squares Identification method uses only a few correlation lag values and understanding the need for decimation, the natural reaction is to include more correlation data into the match to reduce sensitivity and improve noise rejection. However, in the Least Squares Identification formulation this is only possible by increasing the assumed model order, thus introducing spurious modes, whereas the Correlation Fit method allows as many lag values as desired to be used, whilst keeping the assumed model simple.

Results from the Two-Degree-of-Freedom Example

A digitally generated two-degree-of-freedom displacement response to a frequency sweep, including noise, was analyzed using the model matching methods. The damping estimates are presented in Fig. 2 for several numbers of lags in the Correlation Fit method, and with the Least Squares Identification results obtained after decimation by two. For the scope of this note, the results are only intended to show that the Correlation Fit method works since they are only given for one sample of noise! However, indications from other work are that the Correlation Fit method is less sensitive to noise, especially where the segment includes two response peaks.

Conclusions

Two methods for rapid flutter test analysis have been examined. Both are efficient and enable modal frequency and damping estimates to be extracted from the response and excitation time histories. The Correlation Fit method is new and is presented here for the first time. It carries out a least squares fit to as many lag values of the autocorrelation and crosscorrelation functions as desired. The Least Squares Identification method is shown to be a special case of this where only a few lags are used and the fit is exact. The reason for decimation as described in Ref. 1 has been explained.

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Warping of Delta Wings for Minimum Drag

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Nomenclature

AR = aspect ratio
$B(x,y)$ = complete beta function with arguments x and y
C_D = drag coefficient
C_L = lift coefficient
D = drag
e = lift dependent drag factor, $\pi AR \cdot C_D/C_L^2$
L = lift
$s(x)$ = local semispan
$Z(\eta)$ = ordinate distribution of mean camber surface
α_0 = wing incidence
$\epsilon_{on}, \epsilon_n$ = constants in Eq. (1)
δ = wing semiapex angle
η = conical coordinate (Fig. 1)
λ = Lagrange multiplier

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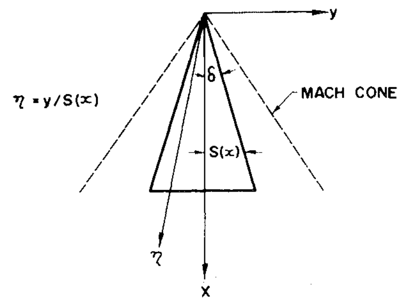

Fig. 1 Wing-Mach cone system.

A LOW aspect ratio delta wing exhibiting an even order polynomial twist distribution was recently reported to give simple expressions for its aerodynamic and geometric properties.[1] Conical camber also has an appealing geometrical simplicity and has found successful applications in, for example, the B-58 Hustler and the Saab 37 Viggen. It was, therefore, thought worthwhile to explore the possibility of using the results of Ref. 1 to design wings of low drag subject to a given lift. This Note summarizes the results of such a study.

The delta wing is placed in a supersonic stream and is well inside the Mach cone, Fig. 1. The linearized equations permit, in such cases, solutions which have either a singularity or zero load at the leading edge. The flow at the leading edge tends to separate with the former type and hence for a smooth flow the latter type is considered by specifying the attachment line to be at the leading edge.[2,3] This produces a leading edge droop which helps alleviate drag.

We now define the nth order basic wing shape as one carrying a downwash distribution of the type

$$\omega_n(\eta) = \epsilon_{on} + \epsilon_n \eta^{2n}/2n \tag{1}$$

where η is the conical coordinate. ϵ_{on} and ϵ_n are related by the attachment condition at the leading edge through

$$\epsilon_{on} = -\frac{\epsilon_n}{2\pi n} B\left(\frac{2n+1}{2}, \frac{1}{2}\right) \tag{2}$$

where B is the complete beta function. It may be shown that the basic shapes have the property that when designed to carry a positive lift, they are at a positive incidence and have a continuous decrease of angle of attack from the wing root to the tip with the tip region showing negative values, Fig. 2a. The basic wings, therefore, will have favorable tip stall characteristics. The incidence distribution gives the mean camber surface a droop at the leading edge which becomes more pronounced with increasing n while at the same time making the central portion of the wing more flat. The surfaces are smooth and do not possess kinks, Fig. 2b.

It is now possible, using the Lagrange multiplier method, to linearly superimpose the basic shapes to obtain wings of low drag for a given lift. Physical observation may then be used to decide on the suitability of a particular design. Mathematically it means that given r basic shapes $N \equiv (n_1, n_2, \ldots, n_r)$ out of the infinite membered set given by Eq. (1) the minimum drag combination subject to a given total lift is obtained by forming the Lagrangian function

$$L \equiv C_D + \lambda C_L = \sum_{i\epsilon N} \sum_{j\epsilon N} C_{D_{i,j}} + \lambda \sum_{i\epsilon N} C_{L_i} \tag{3}$$